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THE STRENGTH OF MEASURABILITY HYPOTHESES

BY

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ABSTRACT

We prove some consequences of various measurability hypotheses. Especially, we establish that the measurability of Σ_2^1 sets implies that Σ_2^1 sets have the property of Baire.

The present paper is devoted to the relationship between measure and category. The modern treatment of both theories has been developed in a completely parallel way (see [3]). This has remained true with metamathematical issues. Solovay has shown the consistency of the statements

(i) every set of reals is Lebesgue measurable,

(ii) every set of reals has the property of Baire,

together with ZF + DC, by considering a single model of set theory ([8]). As is well known, this model was built starting from a model of ZFC with an inaccessible cardinal. Thus, the relative consistency results that could be derived by this method were not consequences of the consistency of ZF alone. The role of the extra hypothesis (the existence of an inaccessible cardinal) has remained unclear until the recent work of Shelah ([7]), which came as a surprise. This role is not the same for measure and category.

THEOREM 1 (Shelah). (i) *If every set of reals is Lebesgue measurable, then $\aleph_1$ is inaccessible in the constructible universe.*

(ii) *Assume there is a standard model of* ZF; *then there is a standard model of* ZF *in which all sets of reals have the property of Baire.*

After this result, a few people (including A. Louveau and the second author) remembered that, at some point, they had considered as a plausible hypothesis

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that measure was stronger than category and, more precisely, that measurability hypotheses concerning some classes of sets could imply analogous statements on the Baire property. Nevertheless, nobody had conjectured the following:

THEOREM 2. *If every Σ^1_2 set of reals is Lebesgue measurable, then every Σ^1_2 subset of 2^ω has the property of Baire.*

REMARKS. (1) As usual 2^ω can be replaced by any polish space in the above result.

(2) The converse of the theorem does not hold; this is proved in [10].

Our proof of Theorem 2 also yields the following:

THEOREM 3. *Let κ be a cardinal. Assume any union of κ many zero-measure sets is of zero-measure. Then*:

(i) *any subset of ω^ω of cardinality κ is σ-bounded,*

(ii) *any union of κ meager sets in 2^ω is meager.*

After we proved these results, we were informed that part (i) of Theorem 3 had been proved independently but earlier by Miller and, similarly, part (ii) by Bartoszynski. After Miller's paper ([2]) became available, we realized that our method of proof could be dualized in order to yield results on the so-called covering property, introduced by Miller.

DEFINITION ([2]). The *covering property* for measure, $C(m)$, is the statement that, for any family of zero-measure sets of power less than the continuum, there is a zero-measure set not covered by any member of the family. The analogous property for category, $C(c)$, is obtained by replacing "zero-measure" by "meager".

THEOREM 4. *$C(c)$ implies $C(m)$.*

Going back to the ideas concerning Theorem 2, it is unclear whether or not measurability hypotheses concerning larger classes of sets have striking consequences. It was tempting to conjecture that the measurability of all subsets of the real line implies that all subsets of 2^ω have the property of Baire and that the perfect set theorem holds, but Shelah has informed us that this conjecture is wrong. We still have a partial result. The MUP (*measurable uniformization property*) is the following property:

(MUP). For any family $(A_x)_{x\in B}$ of non-empty subsets of 2^ω, indexed by the elements of a set B of positive measure, there is a Borel function f, such that

$$\mu(\{x \in B : f(x) \notin A_x\}) = 0$$

(where μ denotes Lebesgue measure).

THEOREM 5. *The* MUP *implies the perfect set theorem.*

Before turning to the proofs, it should be said that all our results use the technique developed by the first author in order to give a direct proof of Shelah's work on the measure problem ([4]) so that the ideas behind the proofs eventually go back to Shelah ([7]).

§1. A combinatorial lemma

1.1. We let $[\omega]^{<\omega}$ denote the set of finite subsets of ω. A mapping $F : \omega \to [\omega]^{<\omega}$ is *slow* if the series $\Sigma_n 1/|F(n)|$ diverges (where $|F(n)|$ is the cardinality of $F(n)$). An element α of ω^ω is eventually covered by F if there exists an integer p such that:

$$\forall n \geqq p \qquad \alpha(n) \in F(n).$$

LEMMA. *Let M be an inner model of* ZFC. *If the union of all Borel sets of zero-measure coded in M is of zero-measure, then there exists a slow application $F : \omega \to [\omega]^{<\omega}$ such that any member of M is eventually covered by F.*

REMARKS. (1) We note that the mapping $n \to \sup F(n)$ eventually dominates all members of $\omega^\omega \cap M$, which is therefore a σ-bounded set.

(2) In his independent work, Bartoszynski has considered a very close combinatorial property. Furthermore, he has (essentially) proved the following converse of Lemma 1. If there is an application F such that

(i) $\forall n \ |F(n)| \leqq n$,

(ii) any member of M is eventually covered by F,

then the union of all Borel sets of zero-measure coded in M is of zero-measure.

(3) Our original version of the lemma ([5]) had a weaker conclusion stating that any member of M is eventually covered by some slow application taken from a fixed sequence (F_n). That our proof could provide the present form of the lemma was pointed out to us by D. H. Fremlin.

1.2. The proof of the lemma stated in the previous section, as well as another subsequent proof, relies on the following basic fact from probability theory, known as the Borel–Cantelli lemma ([6]).

LEMMA. (i) *Let (A_p) be a sequence of events in a probability space. If*

$$\sum_p P(A_p) < \infty$$

then $P(\limsup A_p) = 0$.

(ii) *Let* (A_p) *be a sequence of mutually independent events. If*

$$\sum_p P(A_p) = \infty$$

then $P(\limsup A_p) = 1$.

NOTE. As usual $\limsup A_p$ denotes $\bigcap_n \bigcup_{p \geqq n} A_p$.

1.3. We now turn to the proof of Lemma 1.1. We pick a sequence $A(p,q)$, $p \in \omega$, $q \in \omega$ of mutually independent open subsets of 2^ω of measure $1/p^2$. Actually, as this may be difficult to realize from a technical point of view, we take the measure to be $2^{-2\nu(p)}$ where $\nu(p)$ is the least integer n such that $p < 2^n$. This is easy: $A(p,q)$ is obtained by giving fixed values to a finite set $s(p,q)$ of integers of cardinality $2\nu(p)$. Independence is achieved by taking the sets $s(p,q)$ to be pairwise disjoint. We note that, if p is not zero,

$$\frac{1}{4p^2} \leqq \mu(A(p,q)) \leqq \frac{1}{p^2}.$$

Now, for any $x \in \omega^\omega$, we let

$$G_x = \limsup A(p, x(p));$$

by the Borel–Cantelli lemma, G_x is of zero measure. If M is an inner model satisfying the hypotheses of the lemma, we can find a closed subset B of positive measure, disjoint from $\bigcup_{x \in M} G_x$. We let T be the tree consisting of those sequences of integers s such that

$$\mu(\hat{s} \cap B) > 0,$$

where $\hat{s} = \{\alpha \in 2^\omega : \alpha \text{ extends } s\}$. We note that the set of branches through T is a closed subset B_0 of B with the same measure.

For any pair s, p with $s \in T$ and $p \in \omega$, we let

$$F_s(p) = \{q \in \omega : B_0 \cap \hat{s} \cap A(p,q) = \emptyset\}.$$

CLAIM 1. *$F_s(p)$ is finite.*

PROOF OF CLAIM. Whenever q is in $F_s(p)$, we have

$$B_0 \cap \hat{s} \subseteq 2^\omega - A(p,q).$$

Therefore, because the sets $A(p,q)$ are mutually independent of measure $\geqq 1/4p^2$, we get, if ρ is the cardinality of $F_s(p)$,

$$0<\mu(B_0\cap \hat{s})\leqq\left(1-\frac{1}{4p^2}\right)^{\rho},$$

hence ρ is finite.

CLAIM 2. *F_s is slow.*

PROOF OF CLAIM. We assume s is a fixed element of T and we let

$$\rho(p)=|F_s(p)|.$$

We observe that $B_0\cap\hat{s}$ is disjoint from all sets $A(p,q)$, $p\in\omega$, $q\in F_s(p)$; therefore

$$\limsup_{\substack{p\in\omega\\ q\in F_s(p)}} A(p,q)$$

is not of measure 1. By the Borel–Cantelli lemma, we get

$$\sum_{\substack{p\in\omega\\ q\in F_s(p)}} \mu(A(p,q))<\infty$$

which implies

$$(*)\qquad \sum_p \frac{\rho(p)}{p^2}<\infty.$$

We now set

$$U=\{p:\rho(p)\geqq p\};$$

we get

$$\sum_{p\in U}\frac{\rho(p)}{p^2}<\infty \quad\text{hence}\quad \sum_{p\in U}\frac{1}{p}<\infty$$

so that

$$\sum_{p\notin U}\frac{1}{p}=\infty$$

and therefore

$$\sum_{p\notin U}\frac{1}{\rho(p)}=\infty.$$

CLAIM 3. *There exists a slow mapping $F:\omega\to[\omega]^{<\omega}$ such that, for any $s\in T$, there is an integer n satisfying*

$$\forall p \geqq n \qquad F_s(p)\subseteq F(p).$$

PROOF OF CLAIM. We let

$$F_k(p)=\bigcup_{|s|\leqq k} F_s(p)$$

where $|s|$ denotes the length of the sequence s and

$$\rho_k(p)=|F_k(p)|\leqq \sum_{|s|\leqq k}|F_s(p)|$$

from inequality (*) of Claim 2. It follows that

$$(**) \qquad \sum_p \frac{\rho_k(p)}{p^2}<\infty$$

which, by the same proof as above, gives

$$\sum_p \frac{1}{\rho_k(p)}=\infty.$$

We then define an increasing sequence of integers (n_k) such that, letting

$$I_k=\{p: n_k\leqq k<n_{k+1}\},$$

we have

$$\sum_{p\in I_k}\frac{1}{\rho_k(p)}\geqq 1.$$

By setting $F(p)=F_k(p)$ when $p\in I_k$, we get the desired mapping F.

CLAIM 4. *Any element x of $\omega^\omega\cap M$ is eventually covered by some F_s, hence by F.*

PROOF OF CLAIM. B_0 is disjoint from the G_δ set G_x, hence by the Baire category theorem, one can find $s\in T$ and $j\in\omega$ such that

$$B_0\cap\hat{s}\cap\left(\bigcup_{p\geqq j} A(p,x(p))\right)=\varnothing.$$

This shows that, for $p\geqq j$, we have

$$x(p)\in F_s(p);$$

this finishes the proof.

§2. The main result

In this section, we will assume that the reader is familiar with random (resp. Cohen generic) reals and with the connection between measurability (resp. Baire property) of Σ_2^1 sets and random (resp. Cohen generic) reals. A convenient reference is [9] (inter alia).

2.1. In the next result, the role played by slow mappings appears clearly.

LEMMA. *Let N be an inner model and $F : \omega \to [\omega]^{<\omega}$ be an element of N which is slow. If there is a random real over N, then there exists an element y such that, for any x in $N \cap \omega^\omega$ eventually covered by F, the set*

$$\{p : y(p) = x(p)\}$$

is infinite.

PROOF. We consider the product space

$$\Omega = \prod_{p \in \omega} F(p).$$

Each $F(p)$ is equipped with the equidistributed probability measure and Ω is endowed with the product measure P. Now, it is enough to prove the result for all members of $N \cap \Omega$. If k belongs to $F(p)$, we let

$$A_p^k = \{y : y(p) = k\}.$$

If x is a member of Ω, the sequence $A_p^{x(p)}$ consists of mutually independent events in Ω and we have (because F is slow):

$$\sum_p P(A_p^{x(p)}) = \infty.$$

By the Borel–Cantelli lemma, we get

$$P(\limsup A_p^{x(p)}) = 1.$$

If there is a random real over N, the intersection

$$\bigcap_{x \in N \cap \Omega} \limsup A_p^{x(p)}$$

is not empty and, if y is a member of this intersection, then for any $x \in N \cap \omega^\omega$,

$$\{p : x(p) = y(p)\}$$

is infinite. Hence the lemma is proved.

2.2. We now give the proof of Theorem 2. We will show that, given any inner model $L[\alpha]$ with $\alpha \in \omega^\omega$, there exists a dense G_δ subset of 2^ω, consisting of Cohen generic reals over $L[\alpha]$. This is equivalent to showing that all Σ^1_2 subsets of 2^ω have the property of Baire.

For any nowhere dense closed set C coded in $L[\alpha]$ and any integer n, we let

$$C^{(n)} = \{y \in 2^\omega : \exists x \in C\ \forall p \geqq n\ x(p) = y(p)\};$$

$C^{(n)}$ is nowhere dense. Working in $L[\alpha]$, we let $\theta_C(n)$ be the first integer $k > n$ such that, for some sequence s of length k,

$$\hat{s} \cap C^{(n)} = \emptyset.$$

We note that if $\sigma : [n, \theta_C(n)[\to \omega$ is the restriction of s, then

$$\hat{\sigma} \cap C = \emptyset$$

where $\hat{\sigma} = \{\alpha : \alpha \text{ extends } \sigma\}$.

We now use the fact that the union of all Borel sets of zero measure coded in $L[\alpha]$ is of zero measure, which is a consequence of our hypothesis that Σ^1_2 sets are measurable. By the remark following Lemma 1.1, it follows that some element λ of ω^ω eventually dominates all members of $L[\alpha]$ and therefore all functions $\theta_C(n)$ constructed above. We may assume that λ is strictly increasing and that $\lambda(0)$ is >0 and we define a sequence (u_p) by

$$u_0 = 0,$$

$$u_1 = \lambda(0),$$

$$\dots$$

$$u_{p+1} = \lambda(u_p).$$

By an easy induction, we get

$$\forall p \qquad p < \lambda(p) < u_{p+1}.$$

We now work in the model $M = L[\alpha, \lambda]$. For any closed nowhere-dense set C with a code in $L[\alpha]$ and any integer p, we pick $\sigma_C(p) : [u_p, u_{p+1}[\to \omega$ such that

$$\hat{\sigma}_C(p) \cap C = \emptyset,$$

if such an object exists. We note that θ_C is eventually dominated by λ, so that for p large enough we have

$$u_{p+1} = \lambda(u_p) \geqq \theta_C(u_p),$$

and therefore $\sigma_C(p)$ is defined and is an element of the set Σ of integer-valued functions defined on a finite subset of ω. Identifying Σ with ω and applying once again Lemma 1.1, we get a slow mapping

$$F:\omega \to [\Sigma]^{<\omega}$$

such that, for any nowhere-dense closed set coded in $L[\alpha]$, σ_C is eventually covered by F. We may assume that, for each p, $F(p)$ is a finite set of elements of ω^{I_p}, where I_p is $[u_p, u_{p+1}[$ (by withdrawing the other members of $F(p)$).

We then go over to the model $N = L[\alpha, \lambda, F]$. We know that random reals over N exist (because Σ^1_2 sets are Lebesgue measurable). By Lemma 2.1, identifying once more Σ with ω, we can find $y:\omega \to \Sigma$ such that, for any nowhere-dense closed set C coded in $L[\alpha]$, the set

$$\{p : \sigma_C(p) = y(p)\}$$

is infinite. As above, we may assume that for any p, $y(p) \in \omega^{I_p}$; then, y defines a unique element z of ω^ω such that

$$z \upharpoonright I_p = y(p).$$

Given any nowhere-dense closed set C coded in $L[\alpha]$, the set of p such that

$$\sigma_C(p) = y(p)$$

is infinite, so that the set of integers n such that $z \notin C^{(n)}$ is infinite as well; hence $z \notin \bigcup_n C^{(n)}$.

The last part of the argument takes place in the model $L[\alpha, \lambda, F, z]$. We pick an enumeration (z_n) of all elements of ω^ω which differ from z at finitely many integers only. As z is not a member of $\bigcup_{n\in\omega} C^{(n)}$ for any nowhere-dense closed set C of $L[\alpha]$, we have

$$\forall n \qquad z_n \notin C.$$

We let $q_C(n)$ be an integer such that, letting $t_n = z_n \upharpoonright q_C(n)$, one has

$$\hat{t}_n \cap C = \varnothing.$$

Applying Lemma 1.1 once again, we find an element $r \in \omega^\omega$ eventually dominating all functions q_C. We then define

$$s_n = z_n \upharpoonright r(n).$$

It is easy to check that $\limsup \hat{s}_n$ is a dense G_δ subset of 2^ω consisting of reals which do not belong to any closed nowhere-dense set coded in M. This finishes the proof of Theorem 2.

2.3. We now say a word on Theorem 3. It can be proved by stating the cardinal theoretic analogs of Lemmas 1.1 and 2.1 and imitating the previous proof. For example, the analog of Lemma 1.1 is as follows:

LEMMA. *Let κ be a cardinal strictly less than the continuum. Assume any union of κ-many zero-measure sets is of zero-measure. Then, given any subset X of ω^ω of power κ, there exists a slow application $F:\omega\to[\omega]^{<\omega}$ such that any member of X is eventually covered by F.*

Alternatively, we can prove Theorem 3, by applying Theorem 2 to an inner model $L[A]$, $A \subseteq \kappa$. Such a model contains κ many reals and therefore κ many codes for Borel sets, when κ is regular (or at least when κ is of uncountable cofinality); but the general case can be easily reduced to this special case.

2.4. We now "dualize" the proof carried through in Section 2.2 in order to get the following:

PROPOSITION. *Let M be an inner model. Assume any zero-measure set is covered by some Borel set of zero measure coded in M. Then, any nowhere-dense closed subset of 2^ω is covered by a meager Borel set with a code in M.*

PROOF. We essentially follow the same steps as in the proof of Theorem 2.

STEP 1. *Given any $x\in\omega^\omega$, there exists $F:\omega\to[\omega]^{<\omega}$, which is slow, belongs to M, and is such that*

$$\forall p \qquad x(p)\in F(p).$$

PROOF OF STEP 1. We follow Section 1.3 and we define G_x in exactly the same way. G_x is of zero measure and therefore there exists a closed subset B coded in M and disjoint from G_x. From B, we can define B_0 and the sequence F_s. These definitions take place in M. As in Section 1.3, F_s is slow and for some s and some integer j we have

$$\forall p \geqq j \qquad x(p)\in F_s(p).$$

Modifying F_s for $p<j$, we can realize the stronger property

$$\forall p \qquad x(p)\in F_s(p).$$

STEP 2. *Given any $x\in\omega^\omega$, there exists $y\in\omega^\omega\cap M$ such that*

$$\{p : x(p) = y(p)\}$$

is infinite.

PROOF. We already know that for some slow F in M, $F : \omega \to [\omega]^{<\omega}$, we have

$$\forall p \qquad x(p) \in F(p).$$

We now follow the proof of Lemma 2.1. and define Ω, A_p^k as in this proof. The set $\limsup A_p^{x(p)}$ is of probability one, hence, by the hypothesis, it contains a closed set of positive measure coded in M, hence a member y of M. For this y, we have an infinite set of integers p such that

$$y(p) = x(p).$$

STEP 3. *Any nowhere-dense closed set C is covered by some meager Borel set coded in M.* ·

PROOF. This part is modelled after Section 2.2. We define θ_C as in this section. By Step 1, θ_C is dominated by some element λ of M. We then define from λ a sequence (u_p) as in Section 2.2 and also the analog of σ_C. Applying Step 2, we find an element y of M, $y : \omega \to \Sigma$ such that

(i) $\forall p\ y(p) \in \omega^{I_p}$,

(ii) $\{p : \sigma_C(p) = y(p)\}$ is infinite.

We then consider the real z defined by $z \restriction I_p = y(p)$ and we fix an enumeration (z_n) of all elements of ω^ω which differ from z at finitely many integers only. We define q_C as in the proof of Theorem 2 and, using Step 1 again, we find $r \in M \cap \omega^\omega$, dominating q_C. We conclude the proof by letting $s_n = z_n \restriction r(n)$ and considering $\limsup \hat{s}_n$, which is a dense G_δ with a code in M, whose complement covers the given set C.

2.5. We now consider Theorem 4. We show that $\neg C(m)$ implies $\neg C(c)$. If there is a family of κ many Borel sets $(B_\xi)_{\xi<\kappa}$ of measure zero, such that any zero-measure set is covered by some B_ξ, then we may consider an inner model of type $L[A]$, $A \subseteq \kappa$, in which we can find codes for every Borel set of the sequence (B_ξ). From Proposition 2.3, it follows that any nowhere-dense closed set is covered by some meager Borel set in $L[A]$. If κ is of uncountable cofinality, there are at most κ many such meager sets. If κ is of cofinality ω, we note that Proposition 2.3 still holds when M is not an inner model but the union of an increasing sequence (M_n) of inner models, and we build such a sequence (M_n) with the following properties:

(i) $\bigcup_n M_n$ has κ many reals,

(ii) any Borel set B_ξ has a code in some M_n.

§3. On stronger measurability hypotheses

It was pointed out in the introduction to this paper that it is not clear whether or not measurability hypotheses concerning larger classes of sets have striking consequences. We now prove that the stronger "measurable uniformization property" defined in the introduction implies the perfect set theorem.

3.1. If H is a subset of $\omega^\omega \times 2^\omega \times 2^\omega$, a *section* of H is any subset $H_{\lambda,\alpha}$ of 2^ω, $\lambda \in \omega^\omega$, $\alpha \in 2^\omega$, defined by

$$H_{\lambda,\alpha} = \{\beta : (\lambda, \alpha, \beta) \in H\}.$$

The following is proved (but not actually stated) in the work of the first author on the measure problem ([4]).

Proposition (ZF + DC). *There is a fixed G_δ subset H of $\omega^\omega \times 2^\omega \times 2^\omega$ whose sections are of zero measure and which is such that, for any uncountable subset X of 2^ω,*

(i) *either for some λ, $H_\lambda(X) = \bigcup_{\alpha \in X} H_{\lambda,\alpha}$ is not of zero-measure;*

(ii) *or some filter on ω definable from X is not measurable.*

3.2. We note that the M.U.P. implies the measurability of all sets: in order to prove the measurability of A, one just uniformizes its characteristic function. From this it follows that, given an uncountable subset X of 2^ω, there is an element λ of ω^ω such that $H_\lambda(X)$ is of strictly positive measure. We write H in place of H_λ and H_α in place of $H_{\lambda,\alpha}$. We let B be a Borel set of positive measure included in $H(X)$ and we uniformize the family $(A_\beta)_{\beta \in B}$ where

$$A_\beta = \{\alpha \in X : \beta \in H_\alpha\}.$$

Thus we find a Borel function f such that

$$\mu(\{\beta \in B : f(\beta) \notin A_\beta\}) = 0.$$

We let B_0 be a Borel set of positive measure included in B and such that

$$\forall \beta \in B_0 \qquad f(\beta) \in A_\beta,$$

which means

$$\forall \beta \in B_0 \qquad f(\beta) \in X \;\&\; \beta \in H_{f(\beta)}.$$

We let U be the image of B_0 by f. U is an analytic subset of X, U is not countable, otherwise, we get a contradiction because

$$B_0 \subseteq \bigcup_{\beta \in B_0} H_{f(\beta)} = \bigcup_{\alpha \in U} H_\alpha$$

and each section of H is of zero measure. So U contains a perfect subset which is a subset of X as well. This finishes the proof of Theorem 5.

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